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THE NON-INTEGRABILITY OF A ROTATING ELLIPTICAL BILLIARD†

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The problem of a point moving inside a rotating ellipse is considered; collisions at the boundary are assumed to be absolutely elastic. It is shown that this discrete dynamical system does not admit of an analytic integral independent of the energy integral. The proof of non-integrability is based on the method of separatrix splitting.

Following Birkhoff [1], we consider a dynamical system with elastic reflections describing the motion of a point inside a plane domain D with convex boundary ∂D: inside the domain the point moves uniformly along a straight line; collisions at the boundary are absolutely elastic. In what follows we will call such a system a billiard. As in the case of smooth Hamiltonian systems, one can introduce the concept of an integrable billiard. Together with the energy integral (the velocity of motion is constant), it is sufficient to know one more independent integral. Birkhoff found the integrable case in which the boundary ∂D is an ellipse

$$x^2/a^2+y^2/b^2=1,\quad 0<b^2\leqslant a^2 \tag{0.1}$$

The additional integral [2]

$$F=\dot{x}^2/a^2+\dot{y}^2/b^2-(\dot{x}y-x\dot{y})^2/(a^2b^2) \tag{0.2}$$

is obtained from the Joachimsthal integral of Jacobi's problem of geodesics on the surface of a triaxial ellipsoid [3] by letting one of the axes approach zero. A detailed analysis of the orbits of an elliptical billiard was given by Birkhoff himself [1].

The elliptical Birkhoff billiard seems to be the only integrable billiard with a regular boundary (for a discussion of this conjecture see [1, 2, 4]). On the assumption that the real boundary of the billiard can be continued to a complex curve without singularities, it has been proved [4] that an integral which is a polynomial in the velocity and independent of the energy exists only for an elliptical billiard.

Subsequently [5, 6], this problem was considered in a real setting and conditions were sought for the integrability of perturbed billiards, that is, with the shape of the elliptical boundary slightly varied. The strongest result in this direction has been obtained in [7], where the role of perturbations was taken by boundaries in the shape of an algebraic curve, symmetric about the origin. All these papers corroborate the conjecture that a non-elliptical billiard is not integrable. The non-integrability proofs in [5–7] are based on the separatrix splitting method, discovered by Poincaré.

Perturbing the boundary is not the only way of perturbing an elliptical billiard. For example, one can consider a perturbation in a weak potential field. Integrability has been proved [8, 9] for a billiard in which the point is subject to an elastic force pointing toward the centre of the ellipse or a gravitational force pointing toward one of the foci. These results were generalized in [10].

One further perturbation of an elliptical billiard has been studied [11]: a charged particle moving in a magnetic field of intensity ε orthogonal to the plane of the billiard. Up to terms of order ε^2, the Hamiltonian of this problem is identical to that of the problem of a particle moving inside an ellipse rotating about its centre at low angular velocity $\varepsilon/2$. Numerical computations [11] clearly indicate the stochastization of the orbits for small $\varepsilon\neq 0$.

The purpose of this paper is to give a rigorous proof of the fact that a rotating elliptical billiard is non-integrable for small $\varepsilon\neq 0$.

1. POINCARÉ'S METHOD

Consider a billiard in an ellipse (0.1) which is rotating at low angular velocity ε about its centre. In the intervals between successive impacts the particle moves at a constant velocity along a straight line in a fixed frame of reference. It is convenient to change to a moving system $Oxyz$ rotating

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about the z axis at velocity ε. Relative to the x, y axes, the equation of the boundary of the domain ∂D has the form (0.1). The system has two degrees of freedom; the generalized coordinates are the variables x, y.

Equating the mass of the particle to unity, we write the Lagrangian as

$$L = \tfrac{1}{2}(\dot{x}^2 + \dot{y}^2) + \varepsilon(x\dot{y} - \dot{x}y) + o(\varepsilon) \tag{1.1}$$

The coefficient of ε equals the angular momentum K of the particle relative to the point O. Note that the Lagrangian (1.1) also describes the motion of a charged particle in a weak magnetic field. The non-integrability result established below also holds for that problem.

Let

$$u = \dot{x} - \varepsilon y, \quad v' = \dot{y} + \varepsilon x$$

be the canonical momenta conjugate to the coordinates x, y. Transform from (1.1) to the Hamiltonian

$$H = H_0 + \varepsilon H_1 + o(\varepsilon); \quad H_0 = \tfrac{1}{2}(u^2 + v^2), \quad H_1 = uy - v\,x \tag{1.2}$$

When $a = b$ the perturbed billiard is integrable.

Indeed, the additional integral is the function $F = x\dot{y} - \dot{x}y + \varepsilon(x^2 + y^2)$. It is constant not only on sections of the orbit between impacts, but is also preserved in elastic reflection from the boundary, which, when $a = b$, coincides with a circle with centre at O.

We will show that for $a \neq b$ and small $\varepsilon \neq 0$ a rotating elliptical billiard is a non-integrable dynamical system: it does not admit of a non-constant analytic integral on each level surface of the energy integral $H = \text{const} > 0$. The proof is based on using Poincaré's separatrix splitting method [12] (see also [13, Chap. 5]).

If $a \neq b$, then for every non-zero value of the total energy the unperturbed system has an unstable periodic orbit γ of hyperbolic type: the particle moves along the major axis of the ellipse. The multipliers γ have been calculated [5]; they are λ and $1/\lambda$, where

$$\lambda = (a + \sqrt{a^2 - b^2})/(a - \sqrt{a^2 - b^2}) > 1$$

Since γ is hyperbolic, a family of doubly asymptotic orbits

$$t \to \sigma_\alpha(t), \quad t \in R$$

exists, where α is a parameter that approaches γ_0 indefinitely as $t \to \pm\infty$. As already remarked by Birkhoff [1], the orbits σ_α are made up of segments Δ_n, $n \in Z$, which pass successively through the foci of the ellipse.

We introduce Poincaré's function (throughout what follows, integration will always be performed with respect to t from $-\infty$ to $+\infty$)

$$P(\alpha) = \int H_1(\sigma_\alpha(t))dt \tag{1.3}$$

This improper integral will certainly converge if the integrand tends to zero as $t \to \pm\infty$ (it will then decrease exponentially rapidly as $|t|$ increases), which is equivalent to $H_1(\gamma) = 0$. In the present case this condition is satisfied because $H_1 = \dot{x}y - x\dot{y}$ and $y = 0, \dot{y} = 0$ along the orbit γ.

It is well known [13, Chap. 5] that if $P'(\alpha) \not\equiv 0$, then for small $\varepsilon \neq 0$ the perturbed problem is analytically non-integrable. It can be shown [13] that the function P is periodic in α, and therefore has critical points (for example a maximum and a minimum). It turns out that if the Poincaré function has a non-degenerate critical point, then for small $\varepsilon \neq 0$ the perturbed system has a transverse doubly asymptotic orbit (at a fixed value of the energy), in whose neighbourhood there is a domain of quasi-random (chaotic) behaviour.

In the case under consideration, $H_1 = -K = \text{const}$ on the segments Δ_n. Let us assume (to simplify the notation) that the motion takes place at unit velocity. Then

$$P = -\sum K_n l_n \tag{1.4}$$

where K_n is the angular momentum along the segment Δ_n and l_n is the length of the segment. Here and throughout the rest of the paper, unless otherwise stated, summation is performed from $n = -\infty$ to $n = +\infty$.

2. POINCARÉ'S SERIES

We first observe that the product $K_n l_n$ is equal to $2S_n$, where S_n is the oriented area of the shaded triangle shown in Fig. 1 for the ellipse (0.1): $x = a\cos\varphi, y = b\sin\varphi$. Let $\{\varphi_n\}, n \in Z$ be the sequence of impact points along a doubly asymptotic orbit σ_α. The following formulae are known [5]

$$\varphi_n = 2\operatorname{arctg}\xi_n, \quad \varphi_{n+1} = (2\operatorname{arctg}\xi_{n+1}) + \pi, \quad \xi_n = \lambda^n \operatorname{tg}\alpha, \quad \xi_{n+1} = \lambda\xi_n$$

Consequently

$$2S_n = ab(\cos\varphi_n \sin\varphi_{n+1} - \sin\varphi_n \cos\varphi_{n+1})$$

Using the formulae

$$\cos\varphi_n = \frac{1-\xi_n^2}{1+\xi_n^2}, \quad \sin\varphi_n = \frac{2\xi_n}{1+\xi_n^2}$$

we obtain a formula for the nth term of the series (1.4)

$$\frac{2(\lambda-1)\xi_n(\lambda\xi_n^2+1)}{(\xi_n^2+1)(\lambda^2\xi_n^2+1)}$$

Finally, we obtain an explicit expression for the Poincaré function, expressed as a series

$$P(\alpha) = 2ab(1-\lambda)\sum \frac{\lambda^n \sin\alpha\cos\alpha(\cos^2\alpha + \lambda^{2n+1}\sin^2\alpha)}{(\cos^2\alpha + \lambda^{2n}\sin^2\alpha)(\cos^2\alpha + \lambda^{2n+2}\sin^2\alpha)} \tag{2.1}$$

We will show that $P(\alpha) \not\equiv 0$. Indeed, $P(0) = 0$, but $P(\pi/4) < 0$, since all the terms of the series (2.1) are negative (we are using the fact that $\lambda > 1$). Consequently, by Poincaré's theorem, for small $\varepsilon \neq 0$ a rotating elliptical billiard is a non-integrable dynamical system.

3. CALCULATING THE SUM OF POINCARÉ'S SERIES

Expanding the general term of the series (2.1) as a sum of simple fractions, one can reduce the Poincaré function to the form

$$P(\alpha) = \frac{2ab(1-\lambda)}{1+\lambda}\left[\sum \frac{e^{n\tau}\sin\alpha\cos\alpha}{\cos^2\alpha + e^{2n\tau}\sin^2\alpha} + \sum \frac{\lambda e^{n\tau}\sin\alpha\cos\alpha}{\cos^2\alpha + \lambda^2 e^{2n\tau}\sin^2\alpha}\right]$$

Note that the sums of these two infinite series are equal, since $\lambda = e^{n\tau}$ and n takes all integer values from $-\infty$ to $+\infty$. Therefore, after some reduction, we get

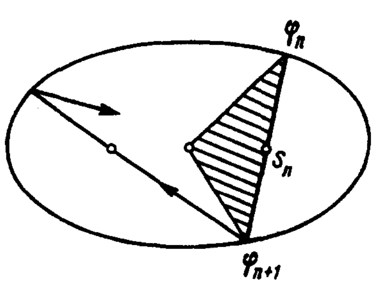

Fig. 1.

$$P(\alpha)=\frac{4ab(1-\lambda)\sin\alpha\cos\alpha}{1+\lambda}\sum f(2\pi n),\quad f(x)=\frac{e^{x\tau/(2\pi)}}{\cos^2\alpha+e^{x\tau/\pi}\sin^2\alpha}$$

If $\sin\alpha = 0$, then $P = 0$. If $\sin\alpha \neq 0$, then $f(x)$ approaches zero at an exponential rate as $|x| \to \infty$, and one can apply the Poisson summation formula [14]

$$\sum f(2\pi n)=\frac{1}{2\pi}\sum\int f(x)e^{-inx}dx$$

These improper integrals can be evaluated by using residues. As a result we obtain the following formula for the Poincaré function

$$P(\alpha)=\frac{4\pi ab(1-\lambda)}{\tau(1+\lambda)}\left[\tfrac{1}{2}+2\sum_{n=1}^{\infty}\frac{e^{-\pi^2 n/\tau}}{1+e^{-2\pi^2 n/\tau}}\cos\left(\frac{n\pi}{\tau}\ln\frac{\cos^2\alpha}{\sin^2\alpha}\right)\right] \tag{3.1}$$

The Poisson summation formula has been used before [7] to simplify Poincaré's series.

We now compare this series with the Fourier series of the Jacobian elliptic function dn (the delta amplitude) [14]

$$\operatorname{dn} z=\frac{\pi}{K}\left[\tfrac{1}{2}+2\sum_{n=1}^{\infty}\frac{q^n}{1+q^{2n}}\cos\left(\frac{n\pi}{K}z\right)\right],\quad q=\exp\left(-\pi\frac{K'}{K}\right)$$

where K is the complete elliptic integral of the second kind and K' is the complete integral with complementary modulus $k' = (1-k^2)^{1/2}$.

We have to put $q = e^{-\pi 2/\tau} < 1$ in (3.1). For this value of one can evaluate the modulus k of the elliptic function and the value of the complete elliptic integral K. After that the Poincaré function (3.1) can be expressed explicitly in terms of the delta amplitude

$$P(\alpha)=\frac{4ab(1-\lambda)K}{\tau(1+\lambda)}\operatorname{dn} z,\quad z=\frac{K}{\tau}\ln\frac{\cos^2\alpha}{\sin^2\alpha}$$

Since $(\operatorname{dn} z)' = -k^2 \operatorname{sn} z \operatorname{cn} z$, the critical points of P are determined from the equalities

$$\frac{1}{\tau}\ln\frac{\cos^2\alpha}{\sin^2\alpha}=m \tag{3.2}$$

where m are integers. Thus, over the interval $(0, \pi)$ there are infinitely many critical values of α (as assumed according to the general theory). Since z is a monotone function of α in the intervals $(0, \pi/2)$ and $(\pi/2, \pi)$, all the critical points are non-degenerate. For $m = 0$ formula (3.2) gives two values: $\alpha = \pi/4$ and $\alpha = 3\pi/4$.

Corresponding to these are two doubly asymptotic orbits of the unperturbed problem which are symmetric about the y axis. At small values of $\varepsilon > 0$ they become transverse doubly asymptotic orbits of the perturbed system, indicating, in particular, the existence of zones of quasi-random motion.

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